ARENS REGULARITY OF BILINEAR FORMS AND UNITAL BANACH MODULE SPACES

KAZEM HAGHNEJAD AZAR AND ABDOLHAMID RIAZI

ABSTRACT. Assume that A, B are Banach algebras and $m : A \times B \to B$, $m' : A \times A \to B$ are bounded bilinear mappings. We will study the relation between Arens regularities of m, m' and the Banach algebras A, B. For Banach $A - bimodule$ B, we show that B factors with respect to A if and only if B^{**} is an unital $A^{**} - module$, and we define locally topological center for elements of A^{**} and will show that when locally topological center of mixed unit of A^{**} is B^{**}, then B^* factors on both sides with respect to A if and only if B^{**} has a unit as $A^{**} - module$.

1. Preliminaries and Introduction

Throughout this paper, A is a Banach algebra and A^*, A^{**}, respectively, are the first and second dual of A. Recall that a left approximate identity ($= LAI$) [resp. right approximate identity ($= RAI$)] in Banach algebra A is a net $(e_\alpha)_{\alpha \in I}$ in A such that $e_\alpha a \longrightarrow a$ [resp. $a e_\alpha \longrightarrow a$]. We say that a net $(e_\alpha)_{\alpha \in I} \subseteq A$ is a approximate identity ($= AI$) for A if it is LAI and RAI for A. If $(e_\alpha)_{\alpha \in I}$ in A is bounded and AI for A, then we say that $(e_\alpha)_{\alpha \in I}$ is a bounded approximate identity ($= BAI$) for A. For $a \in A$ and $a' \in A^*$, we denote by $a'a$ and aa' respectively, the functionals on A^* defined by $< a'a, b > = < a', ab > = a'(ab)$ and $< aa', b > = < a', ba > = a'(ba)$ for all $b \in A$. The Banach algebra A is embedded in its second dual via the identification $< a, a' >$ - $< a', a >$ for every $a \in A$ and $a' \in A^*$. We denote the set $\{a'a : a \in A \text{ and } a' \in A^*\}$ and $\{aa' : a \in A \text{ and } a' \in A^*\}$ by A^*A and AA^*, respectively, clearly these two sets are subsets of A^*.

Let A have a BAI. If the equality $A^*A = A^*$, $(AA^* = A^*)$ holds, then we say that A^* factors on the left (right). If both equalities $A^*A = AA^* = A^*$ hold, then we say that A^* factors on both sides.

The extension of bilinear maps on normed space and the concept of regularity of bilinear maps were studied by [1, 2, 5, 7, 12]. We start by recalling these definitions as follows.

Let X, Y, Z be normed spaces and $m : X \times Y \to Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions m^{***} and m^{t***t} of m from $X^{**} \times Y^{**}$ into Z^{**} as following

1. $m^* : Z^* \times X \to Y^*$, given by $< m^*(z', x), y > = < z', m(x, y) >$ where $x \in X$, $y \in Y$, $z' \in Z^*$,

2. $m^{**} : Y^{**} \times Z^* \to X^*$, given by $< m^{**}(y'', z'), x > = < y'', m^*(z', x) >$ where

$x \in X$, $y'' \in Y^{**}$, $z' \in Z^*$,

3. $m^{***} : X^{**} \times Y^{**} \to Z^{**}$, given by $< m^{***}(x'', y''), z' >$
$=< x'', m^{**}(y'', z') >$ where $x'' \in X^{**}$, $y'' \in Y^{**}$, $z' \in Z^*$.

The mapping m^{***} is the unique extension of m such that $x'' \to m^{***}(x'', y'')$ from X^{**} into Z^{**} is $weak^* - weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $y'' \to m^{***}(x'', y'')$ is not in general $weak^* - weak^*$ continuous from Y^{**} into Z^{**} unless $x'' \in X$. Hence the first topological center of m may be defined as following

$$Z_1(m) = \{x'' \in X^{**} : \ y'' \to m^{***}(x'', y'') \ is \ weak^* - weak^* \ continuous\}.$$

Let now $m^t : Y \times X \to Z$ be the transpose of m defined by $m^t(y, x) = m(x, y)$ for every $x \in X$ and $y \in Y$. Then m^t is a continuous bilinear map from $Y \times X$ to Z, and so it may be extended as above to $m^{t***} : Y^{**} \times X^{**} \to Z^{**}$. The mapping $m^{t***t} : X^{**} \times Y^{**} \to Z^{**}$ in general is not equal to m^{***}, see [1], if $m^{***} = m^{t***t}$, then m is called Arens regular. The mapping $y'' \to m^{t***t}(x'', y'')$ is $weak^* - weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $x'' \to m^{t***t}(x'', y'')$ from X^{**} into Z^{**} is not in general $weak^* - weak^*$ continuous for every $y'' \in Y^{**}$. So we define the second topological center of m as

$$Z_2(m) = \{y'' \in Y^{**} : \ x'' \to m^{t***t}(x'', y'') \ is \ weak^* - weak^* \ continuous\}.$$

It is clear that m is Arens regular if and only if $Z_1(m) = X^{**}$ or $Z_2(m) = Y^{**}$. Arens regularity of m is equivalent to the following

$$\lim_i \lim_j < z', m(x_i, y_j) >= \lim_j \lim_i < z', m(x_i, y_j) >,$$

whenever both limits exist for all bounded sequences $(x_i)_i \subseteq X$, $(y_i)_i \subseteq Y$ and $z' \in Z^*$, see [13].

The mapping m is left strongly Arens irregular if $Z_1(m) = X$ and m is right strongly Arens irregular if $Z_2(m) = Y$.

Let now B be a Banach $A - bimodule$, and let

$$\pi_\ell : \ A \times B \to B \ and \ \pi_r : \ B \times A \to B.$$

be the left and right module actions of A on B, respectively. Then B^{**} is a Banach $A^{**} - bimodule$ with module actions

$$\pi_\ell^{***} : \ A^{**} \times B^{**} \to B^{**} \ and \ \pi_r^{***} : \ B^{**} \times A^{**} \to B^{**}.$$

Similarly, B^{**} is a Banach $A^{**} - bimodule$ with module actions

$$\pi_\ell^{t***t} : \ A^{**} \times B^{**} \to B^{**} \ and \ \pi_r^{t***t} : \ B^{**} \times A^{**} \to B^{**}.$$

We may therefore define the topological centers of the left and right module actions of A on B as follows:

$$Z_{B^{**}}(A^{**}) = Z(\pi_\ell) = \{a'' \in A^{**} : \ the \ map \ b'' \to \pi_\ell^{***}(a'', b'') \ : \ B^{**} \to B^{**}$$
$$is \ weak^* - weak^* \ continuous\}$$
$$Z_{B^{**}}^t(A^{**}) = Z(\pi_r^t) = \{a'' \in A^{**} : \ the \ map \ b'' \to \pi_r^{t***}(a'', b'') \ : \ B^{**} \to B^{**}$$
$$is \ weak^* - weak^* \ continuous\}$$
$$Z_{A^{**}}(B^{**}) = Z(\pi_r) = \{b'' \in B^{**} : \ the \ map \ a'' \to \pi_r^{***}(b'', a'') \ : \ A^{**} \to B^{**}$$

$$is\ weak^* - weak^*\ continuous\}$$

$$Z_{A^{**}}^t(B^{**}) = Z(\pi_\ell^t) = \{b'' \in B^{**} : \ the\ map\ a'' \to \pi_\ell^{t****}(b'', a'') \ : \ A^{**} \to B^{**}$$

$$is\ weak^* - weak^*\ continuous\}$$

We note also that if B is a left(resp. right) Banach $A - module$ and $\pi_\ell : \ A \times B \to B$ (resp. $\pi_r : \ B \times A \to B$) is left (resp. right) module action of A on B, then B^* is a right (resp. left) Banach $A - module$.

We write $ab = \pi_\ell(a, b)$, $ba = \pi_r(b, a)$, $\pi_\ell(a_1 a_2, b) = \pi_\ell(a_1, a_2 b)$,

$\pi_r(b, a_1 a_2) = \pi_r(ba_1, a_2)$, $\pi_\ell^*(a_1 b', a_2) = \pi_\ell^*(b', a_2 a_1)$, $\pi_r^*(b'a, b) = \pi_r^*(b', ab)$,

for all $a_1, a_2, a \in A$, $b \in B$ and $b' \in B^*$ when there is no confusion.

Regarding A as a Banach $A - bimodule$, the operation $\pi : A \times A \to A$ extends to π^{***} and π^{t***t} defined on $A^{**} \times A^{**}$. These extensions are known, respectively, as the first(left) and the second (right) Arens products, and with each of them, the second dual space A^{**} becomes a Banach algebra. In this situation, we shall also simplify our notations. So the first (left) Arens product of $a'', b'' \in A^{**}$ shall be simply indicated by $a''b''$ and defined by the three steps:

$$< a'a, b > = < a', ab >,$$

$$< a''a', a > = < a'', a'a >,$$

$$< a''b'', a' > = < a'', b''a' > .$$

for every $a, b \in A$ and $a' \in A^*$. Similarly, the second (right) Arens product of $a'', b'' \in A^{**}$ shall be indicated by $a''ob''$ and defined by :

$$< aoa', b > = < a', ba >,$$

$$< a'oa'', a > = < a'', aoa' >,$$

$$< a''ob'', a' > = < b'', a'ob'' > .$$

for all $a, b \in A$ and $a' \in A^*$.

The regularity of a normed algebra A is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. Let a'' and b'' be elements of A^{**}, the second dual of A. By *Goldstine's* Theorem [6, P.424-425], there are nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in A such that $a'' = weak^* - \lim_\alpha a_\alpha$ and $b'' = weak^* - \lim_\beta b_\beta$. So it is easy to see that for all $a' \in A^*$,

$$\lim_\alpha \lim_\beta < a', \pi(a_\alpha, b_\beta) > = < a''b'', a' >$$

and

$$\lim_\beta \lim_\alpha < a', \pi(a_\alpha, b_\beta) > = < a''ob'', a' >,$$

where $a''b''$ and $a''ob''$ are the first and second Arens products of A^{**}, respectively, see [5, 11, 12].

We find the usual first and second topological center of A^{**}, which are

$$Z_{A^{**}}(A^{**}) = Z(\pi) = \{a'' \in A^{**} : b'' \to a''b''\ is\ weak^* - weak^*$$

$$continuous\},$$

$$Z_{A^{**}}^t(A^{**}) = Z(\pi^t) = \{a'' \in A^{**} : a'' \to a''ob''\ is\ weak^* - weak^*$$

$$continuous\}.$$

An element e'' of A^{**} is said to be a mixed unit if e'' is a right unit for the first Arens multiplication and a left unit for the second Arens multiplication. That is, e'' is a mixed unit if and only if, for each $a'' \in A^{**}$, $a''e'' = e''oa'' = a''$. By [3, p.146], an element e'' of A^{**} is mixed unit if and only if it is a $weak^*$ cluster point of some BAI $(e_\alpha)_{\alpha \in I}$ in A.

A functional a' in A^* is said to be wap (weakly almost periodic) on A if the mapping $a \to a'a$ from A into A^* is weakly compact. Pym in [13] showed that this definition to the equivalent following condition

For any two net $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $\{a \in A : \| a \| \le 1\}$, we have

$$\lim_\alpha \lim_\beta < a', a_\alpha b_\beta > = \lim_\beta \lim_\alpha < a', a_\alpha b_\beta >,$$

whenever both iterated limits exist. The collection of all wap functionals on A is denoted by $wap(A)$. Also we have $a' \in wap(A)$ if and only if $< a''b'', a' > = < a''ob'', a' >$ for every a'', $b'' \in A^{**}$.

In all of this article, for two normed spaces A and B, $\mathbf{B}(A, B)$ is the set of bounded linear operators from A into B.

In next section, we will study the relationships between the Arens regularity of some bilinear mappings and Banach algebras, that is, for bounded bilinear mappings $m : A \times B \to B$, $m' : A \times A \to B$, if m or m' are Arens regular (resp. irregular), then A or B are Arens regular (resp. irregular), and conversely. We have some applications from this discussion in some of algebra as $L^1(G)$, $M(G)$, $L^\infty(X)$ and $C(X)$ whenever G is a locally compact group and X is a semigroup. As a conclusion, with some conditions, we show that if A or B is not Arens regular, then $A \hat{\otimes} B$ is not Arens regular. In chapter 3, we will extend some problems from [11] into module actions with some new results.

The main results of this paper can be summarized as follows:

a) Let A , B be Banach algebras and B be a Banach $A-bimodule$. Let $T \in \mathbf{B}(A, B)$ be continuous and m be the bilinear mapping from $A \times B$ into B such that for every $a \in A$ and $b \in B$ we have $m(a, b) = T(a)b$. Then we have the following assertions

1) If B is Arens regular, then m is Arens regular.

2) If T is surjective, then we have

i) B is Arens regular if and only if m is Arens regular.

ii) If m is left strongly Arens irregular, then B is left strongly Arens irregular.

iii) If T is injective, then B is left strongly Arens irregular if and only if m is left strongly Arens irregular.

b) Let A , B be Banach algebras and B be a Banach $A-bimodule$. Let $T \in \mathbf{B}(A, B)$ be a homomorphism. If T is weakly compact, then the bilinear mapping $m(a_1, a_2) = T(a_1 a_2)$ from $A \times A$ into B is Arens regular.

c) Assume that B is a right Banach $A-module$ and A^{**} has a right unit as e''. Then, B factors on the right with respect to A if and only if e'' is a right unit $A^{**} - module$ for B^{**}.

d) Assume that A is a Banach algebra and A^{**} has a mixed unit e''. Then we have the following assertions.

i) Let B be a left Banach $A - module$. Then, B^* factors on the left with respect to A if and only if B^{**} has a left unit $A^{**} - module$ as e''.

ii) Let B be a right Banach $A-module$ and $Z_{e''}(\pi_r^t) = B^{**}$. Then, B^* factors on the right with respect to A if and only if B^{**} has a right unit $A^{**} - module$ as e''.

iii) Let B be a Banach $A - bimodule$ and $Z_{e''}(\pi_r^t) = B^{**}$. Then, B^* factors on both sides with respect to A if and only if B^{**} has a unit $A^{**} - module$ as e''.

2. Arens regularity of some bilinear forms

In this part, we introduce some bilinear mappings from $A \times B$ or $A \times A$ into B and make some relations between the Arens regularity of these bilinear mappings and A or B with some applications.

Theorem 1-2. Let A, B be Banach algebras and B be a Banach $A - bimodule$. Let $T \in \mathbf{B}(A, B)$ be continuous and m be the bilinear mapping from $A \times B$ into B such that for every $a \in A$ and $b \in B$ we have $m(a, b) = T(a)b$. Then we have the following assertions

a) If B is Arens regular, then m is Arens regular.

b) If T is surjective, then we have

i) B is Arens regular if and only if m is Arens regular.

ii) If m is left strongly Arens irregular, then B is left strongly Arens irregular.

iii) If T is injective, then B is left strongly Arens irregular if and only if m is left strongly Arens irregular.

Proof. a) By definition of m^{***}, we have $m^{***}(a'', b'') = T^{**}(a'')b''$ and $m^{t***t}(a'', b'') = T^{**}(a'')ob''$ where $a'' \in A^{**}$ and $b'' \in B^{**}$. Since $Z_1(B^{**}) = B^{**}$, the mapping $b'' \to T^{**}(a'')b'' = m^{***}(a'', b'')$ is $weak^* - weak^*$ continuous for all $a'' \in A^{**}$, also since $Z_2(B^{**}) = B^{**}$, the mapping $a'' \to T^{**}(a'')ob'' = m^{t***t}(a'', b'')$ is $weak^* - weak^*$ continuous for all $b'' \in A^{**}$. Hence m is Arens regular.

b) i) Let m be Arens regular. Then $Z_1(m) = A^{**}$ and $Z_2(m) = B^{**}$. Let $b_1'', b_2'' \in B^{**}$ and $(b_\alpha'')_\alpha \in B^{**}$ such that $b_\alpha'' \overset{w^*}{\to} b_2''$. Assume that $a'' \in A^{**}$ such that $T^{**}(a'') = b_1''$. Then we have

$$b_1'' b_2'' = T^{**}(a'')b_2'' = m^{***}(a'', b_2'') = weak^* - \lim_\alpha m^{***}(a'', b_\alpha'')$$

$$= weak^* - \lim_\alpha T^{**}(a'')b_\alpha'' = weak^* - \lim_\alpha b_1'', b_\alpha''.$$

Hence $Z_1(B^{**}) = B^{**}$ consequently B is Arens regular.

b) ii) Let m be left strongly Arens irregular then $Z_1(m) = A$. For $b_1'' \in Z_1(B^{**})$ the mapping $b_2'' \to b_1'' b_2''$ is $weak^* - weak^*$ continuous. Also since T is surjective, there exists $a'' \in A^{**}$ such that $T^{**}(a'') = b_1''$ and the mapping $b_2'' \to T^{**}(a'')b_2'' = m^{***}(a'', b_2'')$ is $weak^* - weak^*$ continuous. Hence $a'' \in Z_1(m) = A$. Consequently we have $b_1'' = T^{**}(a'') \in B$. It follows that $Z_1(B^{**}) = B$.

b) iii) Let B be left strongly Arens irregular, so $Z_1(B^{**}) = B$. For $a'' \in Z_1(m)$ the mapping $b'' \to m^{***}(a'', b'')$ is $weak^* - weak^*$ continuous consequently $T^{**}(a'') \in Z_1(B^{**}) = B$. Since T is bijective, $a'' \in A$. Hence we conclude $Z_1(m) = A$. $\square$

In Theorem 1-2, if we replace the left strongly Arens irregularity of A, B and m with right strongly Arens irregularity of them, then the results will be similar.

The following definition which introduced by Ulger [17] has an important role in showing some sufficient condition for the Arens regularity of tensor product $A\hat{\otimes}B$ where A and B are Banach algebra.
We recall that a bilinear form $m : A \times B \to C$ is biregular if for any two pairs of sequence $(a_i)_i$, $(\tilde{a}_j)_j$ in A_1 and $(b_i)_i$, $(\tilde{b}_j)_j$ in B_1, we have

$$\lim_i \lim_j m(a_i\tilde{a}_j, b_i\tilde{b}_j) = \lim_j \lim_i m(a_i\tilde{a}_j, b_i\tilde{b}_j)$$

provided that these limits exist.

Corollary 2-2. Let B be an unital Banach algebra and suppose that A is subalgebra of B. If A is not Arens regular, then $A\hat{\otimes}B$ is not Arens regular.

Proof. Let $m : A \times B \to C$ be the bilinear form that introduced in Theorem 1-2 where $T : A \to B$ is natural inclusion. Since A is not Arens regular, m is not biregular. Consequently by [17, Theorem 3.4], $A\hat{\otimes}B$ is not Arens regular. $\square$

Example 3-2. Let $X = [0, 1]$ be the unit interval and let $C(X)$ be the Banach algebra of all continuous bounded functions on X with supremum norm and the convolution as multiplication defined by

$$f * g(x) = \int_0^x f(x - t)g(t)dt \ where \ 0 \le x \le 1.$$

Let $T : C(X) \to L^\infty(X)$ be the natural inclusion and $m : C(X) \times L^\infty(X) \to L^\infty(X)$ be defined by $m(f, g) = f * g$ where $f \in C(X)$ and $g \in L^\infty(X)$. By [2], $L^\infty(X)$ is Arens regular and by Theorem 1-2, we conclude that m is Arens regular.
Similarly since c_0 is Arens regular, see[1, 5], by using Theorem 1-2, we can show that the bounded bilinear mapping $(f, g) \to f * g$ from $\ell^1 \times c_0$ into c_0 is Arens regular.

For a Banach algebra A, we recall that a bounded linear operator $T : A \to A$ is said to be a left (resp. right) multiplier if, for all $a, b \in A$, $T(ab) = T(a)b$ (resp. $T(ab) = aT(b)$). We denote by $LM(A)$ (resp. $RM(A)$) the set of all left (resp. right) multipliers of A. The set $LM(A)$ (resp. $RM(A)$) is normed subalgebra of the algebra $L(A)$ of bounded linear operator on A.
Now, we define a new concept as follows which is an extension of Left [right] multiplier on a Banach algebra. We will show some relation between this concept and Arens regularity of some bilinear mappings in Theorem 6-2.

Definition 4-2. Let B be a left Banach [resp. right] $A - module$ and $T \in \mathbf{B}(A, B)$. Then T is called extended left [resp. right] multiplier if $T(a_1a_2) = \pi_r(T(a_1), a_2)$ [*resp.* $T(a_1a_2) = \pi_\ell(a_1, T(a_2))$] for all $a_1, a_2 \in A$.
We show by $LM(A, B)$ [*resp.* $RM(A, B)$] all of the Left [resp. right] multiplier extension from A into B.

Example 5-2. Let $a' \in A^*$. Then the mapping $T_{a'} : a \to a'a$ [*resp.* $R_{a'} \ a \to aa'$] from A into A^* is left [right] multiplier, that is, $T_{a'} \in LM(A, A^*)$ [$R_{a'} \in RM(A, A^*)$]. $T_{a'}$ is weakly compact if and only if $a' \in wap(A)$. So, we can write $wap(A)$ as a subspace of $LM(A, A^*)$.

Theorem 6-2. Let B be a left Banach $A - module$ and $T \in \mathbf{B}(A, B)$ be a continuous map. Assume that $m : A \times A \to B$ is the bilinear mapping such that $m(a_1, a_2) = T(a_1 a_2)$. Then we have the following assertions

i) If A is Arens regular, then m is Arens regular.

ii) If m is left [right] strongly Arens irregular, then A is left [right] strongly Arens irregular.

iii) $T^{**}(Z_1(m)) \subseteq Z_{A^{**}}(B^{**})$.

iv) If $T \in LM(A, B)$, then $T^{**} \in LM(A^{**}, B^{**})$.

v) Suppose that B is Banach algebra and T is epimorphism. Then, B is Arens regular if and only if m is Arens regular.

Proof. i) An easy calculation shows that

$$m^{***}(a_1'', a_2'') = T^{**}(a_1'' a_2'') \ , \ m^{t***t}(a_1'', a_2'') = T^{**}(a_1'' o a_2'').$$

Since A is Arens regular, the mapping $a_2'' \to a_1'' a_2''$ is $weak^* - weak^*$ continuous for all $a_1'' \in A^{**}$. Also the mapping $a_1'' \to a_1'' o a_2''$ is $weak^* - weak^*$ continuous for all $a_2'' \in A^{**}$. Hence both mappings $a_2'' \to T^{**}(a_1'' a_2'') = m^{***}(a_1'', a_2'')$ and $a_1'' \to T^{**}(a_1'' o a_2'') = m^{t***t}(a_1'', a_2'')$ are $weak^* - weak^*$ continuous for all $a_1'' \in A^{**}$ and $a_2'' \in A^{**}$, respectively. We conclude that $Z_1(m) = Z_2(m) = A^{**}$.

ii) Let $a_1'' \in Z_1(A^{**})$. Then the mapping $a_2'' \to a_1'' a_2''$ is $weak^* - weak^*$ continuous consequently the mapping $a_2'' \to T^{**}(a_1'' a_2'') = m^{***}(a_1'', a_2'')$ is $weak^* - weak^*$ continuous. Hence $a_1'' \in Z_1(m) = A$.

iii) Let $a_1'' \in Z_1(m)$. Then the mapping

$$a_2'' \to m^{***}(a_1'', a_2'') = T^{**}(a_1'') a_2''$$

is $weak^* - weak^*$ continuous from A^{**} into B^{**}. It follows that $T^{**}(a_1'') \in Z_{A^{**}}(B^{**})$.

iv) If we set $m(a_1, a_2) = T(a_1 a_2)[resp. = T(a_1)a_2]$ for all $a_1, a_2 \in A$, then $m^{***}(a_1'', a_2'') = T^{**}(a_1'' a_2'')$ $[resp. = T^{**}(a_1'') a_2'']$ for all $a_1'', a_2'' \in A^{**}$. Thus, we conclude that $T^{**}(a_1'' a_2'') = T^{**}(a_1'') a_2''$ for all $a_1'', a_2'' \in A^{**}$.

v) Let m be Arens regular and $b_1'', b_2'' \in B^{**}$ and let $(b_\alpha'')_\alpha \in B^{**}$ such that $b_\alpha'' \overset{w^*}{\to} b_2''$. We set $a_1'', a_2'' \in A^{**}$ and $(a_\alpha'')_\alpha \in A^{**}$ such that $T^{**}(a_1'') = b_1''$, $T^{**}(a_2'') = b_2''$ and $T^{**}(a_\alpha'') = b_\alpha''$. Then

$$b_1'' b_2'' = T^{**}(a_1'') T^{**}(a_2'') = T^{**}(a_1'' a_2'') = m^{***}(a_1'', a_2'')$$

$$= weak^* - \lim_\alpha m^{***}(a_1'', a_\alpha'') = weak^* - \lim_\alpha T^{**}(a_1'' a_\alpha'')$$

$$= weak^* - \lim_\alpha T^{**}(a_1'') T^{**}(a_\alpha'') = weak^* - \lim_\alpha b_1'' b_\alpha'',$$

where by the open mapping Theorem, we have $a_\alpha'' \overset{w^*}{\to} a_2''$. Consequently $Z_1(B^{**}) = B^{**}$.

Conversely, let B be Arens regular and $a_1'', a_2'' \in A^{**}$ and $(a_\alpha)_\alpha \in A^{**}$ such that $a_\alpha'' \overset{w^*}{\to} a_2''$. Then

$$m^{***}(a_1'', a_2'') = T^{**}(a_1'' a_2'') = weak^* - \lim_\alpha T^{**}(a_1'' a_\alpha'')$$

$$= weak^* - \lim_\alpha m^{***}(a_1'', a_\alpha'').$$

It follow that $Z_1(m) = A^{**}$. Thus m is Arens regular. □

Example 7-2. Assume that $T : c_0 \to \ell^\infty$ is the natural inclusion and $m : c_0 \times c_0 \to \ell^\infty$ be the bilinear mapping such that $m(f, g) = f * g$. Since c_0 is Arens regular, then m is Arens regular. Similarly the bilinear mapping $m : C(G) \times C(G) \to L^\infty(G)$ defined by formula $(f, g) \to f * g$ is Arens regular whenever G is compact.

For normed spaces X, Y, Z, W, let $m_1 : X \times Y \to Z$ and $m_2 : X \times W \to Z$ be bounded bilinear mappings. If $h : Y \to W$ is a continuous linear mapping such that $m_1(x, y) = m_2(x, h(y))$ for all $x \in X$ and $y \in Y$, then we say that m_1 factors through m_2, see[2]. We say that the continuous bilinear mapping $m : X \times Y \to Z$ factors if m is onto Z, see [7].

Theorem 8-2. Let A , B be Banach algebras and B be a Banach $A - bimodule$. Let $T \in \mathbf{B}(A, B)$ be a continuous homomorphism. If T is weakly compact, then the bilinear mapping $m(a_1, a_2) = T(a_1 a_2)$ from $A \times A$ into B is Arens regular.

Proof. Let m' be the bilinear mapping that we introduced in Theorem 1-2. Then $m(a_1, a_2) = m'(a_1, Ta_2)$ for all $a_1, a_2 \in A$. Consequently m factors through m', so by [2, Theorem 2], we conclude that m is Arens regular.

Example 9-2. Suppose that $T : L^1(G) \to M(G)$ is the natural inclusion. Then the bilinear mapping $m : L^1(G) \times L^1(G) \to M(G)$ defined by $m(f, g) = f * g$ for all $f, g \in L^1(G)$ is Arens regular whenever G is finite, see [14]. Also the left strongly Arens irregularity of m implies that $L^1(G)$ is also left strongly Arens irregular, see [10, 11]. $\square$

3. Unital $A - modules$ and module actions

In [11], Lau and Ulger show that for Banach algebra A, A^* factors on the left if and only if A^{**} is unital with respect to the first Arens product. In this chapter we extend this problem to module actions with some results.

We say that A^{**} has a *weak* bounded left approximate identity*$(= W^*BLAI)$ with respect to the first Arens product, if there is a bounded net as $(e_\alpha)_\alpha \subseteq A$ such that for all $a'' \in A^{**}$ and $a' \in A^*$, we have $< e_\alpha a'', a' > \to < a'', a' >$. The definition of W^*RBAI is similar to W^*LBAI and if A^{**} has both W^*LBAI and W^*RBAI, then we say that A^{**} has W^*BAI.

Assume that B is a Banach $A - bimodule$. We say that B factors on the left (right) with respect to A if $B = BA$ ($B = AB$). We say that B factors on both sides, if $B = BA = AB$.

Definition 1-3. Let B be a left Banach $A - module$ and e be a left unit element of A. Then we say that e is a left unit (resp. weakly left unit) $A - module$ for B, if $\pi_\ell(e, b) = b$ (resp. $< b', \pi_\ell(e, b) > = < b', b >$ for all $b' \in B^*$) where $b \in B$. The definition of right unit (resp. weakly right unit) $A - module$ is similar.

We say that a Banach $A - bimodule$ B is a unital $A - module$, if B has left and right unit $A - module$ that are equal then we say that B is unital $A - module$.

Let B be a left Banach $A - module$ and $(e_\alpha)_\alpha \subseteq A$ be a LAI [resp. weakly left approximate identity(=WLAI)] for A. We say that $(e_\alpha)_\alpha$ is left approximate identity

$(= LAI)[$ resp. weakly left approximate identity $(=WLAI)]$ for B, if for all $b \in B$, we have $\pi_\ell(e_\alpha, b) \to b$ (resp. $\pi_\ell(e_\alpha, b) \xrightarrow{w} b)$. The definition of the right approximate identity $(= RAI)[$ resp. weakly right approximate identity $(= WRAI)]$ is similar.

We say that $(e_\alpha)_\alpha$ is a approximate identity $(= AI)[$ resp. weakly approximate identity $(WAI)]$ for B, if B has left and right approximate identity [resp. weakly left and right approximate identity] that are equal.

Let $(e_\alpha)_\alpha \subseteq A$ be $weak^*$ left approximate identity for A^{**}. We say that $(e_\alpha)_\alpha$ is $weak^*$ left approximate identity $A^{**} - module$ $(= W^*LAI$ $A^{**} - module)$ for B^{**}, if for all $b'' \in B^{**}$, we have $\pi_\ell^{***}(e_\alpha, b'') \xrightarrow{w^*} b''$. The definition of the $weak^*$ right approximate identity $A^{**} - module(= W^*RAI$ $A^{**} - module)$ is similar.

We say that $(e_\alpha)_\alpha$ is a $weak^*$ approximate identity $A^{**} - module$ $(= W^*AI$ $A^{**} - module)$ for B^{**}, if B^{**} has $weak^*$ left and right approximate identity $A^{**} - module$ that are equal.

Example 2-3. i) $L^1(G)$ is a Banach $M(G) - bimodule$ under convolution as multiplication. It is clear that $L^1(G)$ is a $unital$ $M(G) - bimudole$.

ii) Since $L^p(G)$, for $1 \le p < \infty$, is a left Banach $M(G) - module$, by using [8, 10.15], $L^p(G)$ has a $BLAI$ $(e_\alpha)_\alpha \subset M(G)$.

Theorem 3-3. Assume that A is a Banach algebra and has a BAI $(e_\alpha)_\alpha$. Then we have the following assertions.

i) Let B be a right Banach $A - module$. Then B factors on the left with respect to A if and only if B has a $WRAI$.

ii) Let B be a left Banach $A - module$. Then B factors on the right with respect to A if and only if B has a $WLAI$.

iii) B factors on both side with respect to A if and only if B has a WAI.

Proof. i) Suppose that $B = BA$. Let $b \in B$ and $b' \in B^*$ then there are $x \in B$ and $a \in A$ such that $b = xa$. Then

$$< b', \pi_r(b, e_\alpha) >=< b', \pi_r(xa, e_\alpha) >=< \pi_r^*(b', x), ae_\alpha >\to< \pi_r^*(b', x), a >$$

$$=< b', \pi_r(x, a) >=< b', b > .$$

It follows that $\pi_r(b, e_\alpha) \xrightarrow{w} b$ consequently B has a $WRAI$.

For the converse, since BA is a weakly closed subspace of B, so by Cohen Factorization theorem, see [5], proof is hold.

ii) The proof is similar to (i).

iii) Clear. $\qquad\qquad\square$

In Theorem 3-3, if we set $B = A$, then we obtain Lemma 2.1 from [11].

Theorem 4-3. Assume that B is a right Banach $A - module$ and A^{**} has a right unit as e''. Then, B factors on the right with respect to A if and only if e'' is a right unit $A^{**} - module$ for B^{**}.

Proof. Since A^{**} has a right unit e'', there is a $BRAI$ $(e_\alpha)_\alpha$ for A such that $e_\alpha \xrightarrow{w^*} e''$. Let $AB = B$ and $b \in B$. Thus, there is $x \in B$ and $a \in A$ such that $b = ax$. Then for

all $b' \in B^*$, we have

$$< \pi_r^{**}(e'', b'), b >=< e'', \pi_r^*(b', b) >= \lim_\alpha < e_\alpha, \pi_r^*(b', b) >$$

$$= \lim_\alpha < \pi_r^*(b', b), e_\alpha >= \lim_\alpha < b', \pi_r(b, e_\alpha) >$$

$$= \lim_\alpha < b', \pi_r(xa, e_\alpha) >= \lim_\alpha < \pi_r^*(b', x), ae_\alpha >$$

$$=< \pi_r^*(b', x), a >=< b', \pi_r(x, a) =< b', b > .$$

Thus $\pi_r^{**}(e'', b') = b'$. Now let $b'' \in B^{**}$, then we have

$$< \pi_r^{***}(b'', e''), b' >=< b'', \pi_r^{**}(e'', b') >=< b'', b' > .$$

We conclude that $\pi_r^{***}(b'', e'') = b''$. Hence it follows that B^{**} has a right unit $A^{**} - module$.

Conversely, assume that e'' is a right unit $A^{**} - module$ for B^{**}. Let $b \in B$ and $b' \in B$. Then we have

$$< b', \pi_r(b, e_\alpha) >=< \pi_r(b', b), e_\alpha) > \rightarrow < \pi_r(b', b), e'' >=< b', \pi_r(b, e'') >$$

$$=< b', b > .$$

Consequently $\pi_r(b, e_\alpha) \overset{w}{\rightarrow} \pi_r(b, e'') = b$, it follows that $b \in \overline{BA}^w$. Since BA is a *weakly* closed subspace of B, so by Cohen Factorization theorem, $b \in BA$. $\qquad \square$

Definition 5-3. Let B be a Banach $A - bimodule$ and $a'' \in A^{**}$. We define the locally topological center of the left and right module actions of a'' on B, respectively, as follows

$$Z_{a''}^t(B^{**}) = Z_{a''}^t(\pi_\ell^t) = \{b'' \in B^{**} : \pi_\ell^{t***t}(a'', b'') = \pi_\ell^{***}(a'', b'')\},$$

$$Z_{a''}(B^{**}) = Z_{a''}(\pi_r^t) = \{b'' \in B^{**} : \pi_r^{t***t}(b'', a'') = \pi_r^{***}(b'', a'')\}.$$

It is clear that

$$\bigcap_{a'' \in A^{**}} Z_{a''}^t(B^{**}) = Z_{A^{**}}^t(B^{**}) = Z(\pi_\ell^t),$$

$$\bigcap_{a'' \in A^{**}} Z_{a''}(B^{**}) = Z_{A^{**}}(B^{**}) = Z(\pi_r).$$

Theorem 6-3. Assume that A is a Banach algebra and A^{**} has a mixed unit e''. Then we have the following assertions.

i) Let B be a left Banach $A - module$. Then, B^* factors on the left with respect to A if and only if B^{**} has a left unit $A^{**} - module$ as e''.

ii) Let B be a right Banach $A - module$ and $Z_{e''}(\pi_r^t) = B^{**}$. Then, B^* factors on the right with respect to A if and only if B^{**} has a right unit $A^{**} - module$ as e''.

iii) Let B be a Banach $A - bimodule$ and $Z_{e''}(\pi_r^t) = B^{**}$. Then, B^* factors on both sides with respect to A if and only if B^{**} has a unit $A^{**} - module$ as e''.

Proof. i) Let $(e_\alpha)_\alpha \subseteq A$ be a BAI for A such that $e_\alpha \xrightarrow{w^*} e''$. Suppose that $B^*A = B^*$. Thus for all $b' \in B^*$ there are $a \in A$ and $x' \in B^*$ such that $x'a = b'$. Then for all $b'' \in B^{**}$ we have

$$< \pi_\ell^{***}(e'', b''), b' >=< e'', \pi_\ell^{**}(b'', b') >= \lim_\alpha < \pi_\ell^{**}(b'', b'), e_\alpha >$$

$$= \lim_\alpha < b'', \pi_\ell^*(b', e_\alpha) >= \lim_\alpha < b'', \pi_\ell^*(x'a, e_\alpha) >$$

$$= \lim_\alpha < b'', \pi_\ell^*(x', ae_\alpha) >= \lim_\alpha < \pi_\ell^{**}(b'', x'), ae_\alpha >$$

$$=< \pi_\ell^{**}(b'', x'), a >=< \pi_\ell^{***}(b'', b' > .$$

Thus $\pi_\ell^{***}(e'', b'') = b''$ consequently B^{**} has left unit $A^{**} - module$.

Conversely, Let e'' be a left unit $A^{**} - module$ for B^{**} and $b' \in B^*$. Then for all $b'' \in B^{**}$ we have

$$< b'', b' >=< \pi_\ell^{***}(e'', b''), b' >=< e'', \pi_\ell^{**}(b'', b') >$$

$$= \lim_\alpha < \pi_\ell^{**}(b'', b'), e_\alpha >= \lim_\alpha < b'', \pi_\ell^*(b', e_\alpha) > .$$

Thus we conclude that $weak - \lim_\alpha \pi_\ell^*(b', e_\alpha) = b'$. So by Cohen Factorization theorem, we are done.

ii) Suppose that $AB^* = B^*$. Thus for all $b' \in B^*$ there are $a \in A$ and $x' \in B^*$ such that $ax' = b'$. Assume $(e_\alpha)_\alpha \subseteq A$ is a BAI for A such that $e_\alpha \xrightarrow{w^*} e''$. Let $b'' \in B^{**}$ and $(b_\beta)_\beta \subseteq B$ such that $b_\beta \xrightarrow{w^*} b''$. Then

$$< \pi_r^{***}(b'', e''), b' >= \lim_\beta < \pi_r^{***}(b_\beta, e''), b' >= \lim_\beta \lim_\alpha < b', \pi_r(b_\beta, e_\alpha) >$$

$$= \lim_\beta \lim_\alpha < ax', \pi_r(b_\beta, e_\alpha) >= \lim_\beta \lim_\alpha < x', \pi_r(b_\beta, e_\alpha)a >$$

$$= \lim_\beta \lim_\alpha < x', \pi_r(b_\beta, e_\alpha a) >= \lim_\beta \lim_\alpha < \pi_r^*(x', b_\beta), e_\alpha a >$$

$$= \lim_\beta < \pi_r^*(x', b_\beta), a >=< b'', b' > .$$

We conclude that

$$\pi_r^{***}(b'', e'') = b''$$

for all $b'' \in B^{**}$.

Conversely, Suppose that $\pi_r^{***}(b'', e'') = b''$ where $b'' \in B^{**}$ and $(b_\beta)_\beta \subseteq B$ such that $b_\beta \xrightarrow{w^*} b''$. Let $(e_\alpha)_\alpha \subseteq A$ be a BAI for A such that $e_\alpha \xrightarrow{w^*} e''$. Then for all $b' \in B^*$ we have

$$< b'', b' >=< \pi_r^{***}(b'', e''), b' >=< b'', \pi_r^{**}(e'', b') >= \lim_\beta < \pi_r^{**}(e'', b'), b_\beta >$$

$$= \lim_\beta < e'', \pi_r^*(b', b_\beta) >= \lim_\beta \lim_\alpha < \pi_r^*(b', b_\beta), e_\alpha >$$

$$= \lim_\beta \lim_\alpha < \pi_r^*(b', b_\beta), e_\alpha >= \lim_\beta \lim_\alpha < b', \pi_r(b_\beta, e_\alpha) >$$

$$= \lim_\alpha \lim_\beta < \pi_r^{***}(b_\beta, e_\alpha), b' >= \lim_\alpha \lim_\beta < b_\beta, \pi_r^{**}(e_\alpha, b') >$$

$$= \lim_\alpha < b'', \pi_r^{**}(e_\alpha, b') > .$$

It follows that $weak - \lim_\alpha \pi_r^{**}(e_\alpha, b') = b'$. So by Cohen Factorization Theorem, we are done.

iii) Clear. $\qquad\square$

Corollary 7-3. Let B be a Banach $A - bimodule$ and A^{**} has a mixed unit e''.

a) Let $Z_{e''}(\pi_r^t) = B^{**}$. Then we have the following assertions

i) If B or B^* factors on the right but not on the left with respect to A then $\pi_\ell \neq \pi_r^t$.

ii) If B^* factors on the left with respect to A and $\pi_\ell = \pi_r^t$, then B^* factors on the right with respect to A.

iii) e'' is a right unit $A^{**}-module$ for B^{**} if and only if $(e_\alpha)_\alpha$ is a $W^*RAI\ A^{**}-module$ for B^{**} whenever $e_\alpha \xrightarrow{w^*} e''$.

b) Let $Z_{e''}^t(\pi_\ell^t) = B^{**}$. Then we have the following assertions

i) If B or B^* factors on the right but not on the left with respect to A then $\pi_r \neq \pi_\ell^t$.

ii) If B^* factors on the right with respect to A and $\pi_r = \pi_\ell^t$, then B^* factors on the left with respect to A.

iii) e'' is a left unit $A^{**}-module$ for B^{**} if and only if $(e_\alpha)_\alpha$ is a $W^*LAI\ A^{**}-module$ for B^{**} whenever $e_\alpha \xrightarrow{w^*} e''$.

c) Let $Z_{e''}^t(\pi_\ell^t) = Z_{e''}(\pi_r^t) = B^{**}$. Then we have the following assertions

i) If B^* not factors on the right and left with respect to A then $\pi_r \neq \pi_\ell^t$ and $\pi_\ell \neq \pi_r^t$.

ii) e'' is a unit $A^{**} - module$ for B^{**} if and only if $(e_\alpha)_\alpha$ is a $W^*AI\ A^{**} - module$ for B^{**} whenever $e_\alpha \xrightarrow{w^*} e''$.

Proof. a) i) Let B or B^* factors on the right but not on the left with respect to A. By Theorem 4-3 (resp. Theorem 5-3), e'' is a right unit $A^{**} - module$ for B^{**}. Thus we have $\pi_r^{***}(b'', e'') = b''$ for all $b'' \in B^{**}$. If we set $\pi_\ell = \pi_r^t$, then $\pi_\ell^{***}(e'', b'') = \pi_r^{t***}(e'', b'') = \pi_r^{t***t}(b'', e'') = \pi_r^{***}(b'', e'') = b''$ for all $b'' \in B^{**}$. Consequently e'' is left unit $A^{**} - module$ for B^{**}. Then by Theorem 4-3 (resp. Theorem 5-3), B or B^* factors on the left which is impossible.

ii) Similar to (i).

iii) Since $e_\alpha \xrightarrow{w^*} e''$, $weak^* - \lim_\alpha \pi_r^{***}(b'', e_\alpha) = \pi_r^{***}(b'', e'')$ for all $b'' \in B^{**}$ hence we are done.

The proofs of (b) and (c) is the same and easy. $\qquad\square$

Assume that $Z_{e''}^t(\pi_\ell^t) = Z_{e''}(\pi_r^t) = B^{**}$. Let $\pi_r = \pi_\ell^t$ and $\pi_\ell = \pi_r^t$. We conclude from Corollary 7-3 that if B^* also factors on the one side, then B^* factors on the other side.

In Theorem 7-3. if we set $B = A$, then we obtain the Proposition 2.10 from [11].

Problems.

Suppose that B is a Banach $A-bimodule$. Which condition we need for the following assertions

i) B factors on the left with respect to A if and only if B^{**} has a left unit $A^{**}-module$.

ii) B factors on the one side with respect to A if and only if B^* factors on the same side with respect to A.

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Department of Mathematics, Amirkabir University of Technology, Tehran, Iran
Email address: haghnejad@aut.ac.ir

Department of Mathematics, Amirkabir University of Technology, Tehran, Iran
Email address: riazi@aut.ac.ir